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The centre and isochronicity problems for some cubic systems

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Abstract

We present an efficient method for computing focus and linearizability quantities of polynomial differential equation systems. We apply the method to computing these quantities for ten eight-parametric cubic systems and obtain the necessary and sufficient conditions of linearizability (isochronicity) of these systems. We also show that there is a kind of duality between the problem of constructing algebraic invariant curves, first integrals and linearizing transformations on one side, and the problem of solving some first-order linear partial differential equations on the other side.

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1. Introduction

The problem of integrability for differential equation systems is one of the major mathematical problems and is of great importance for various applications. A significant area of study concerns the investigation of local first integrals. In the case of real systems of differential equations of the form

$$\dot{u} = -v + U(u, v) \qquad \dot{v} = u + V(u, v) \tag{1}$$

(where $U(u, v)$, $V(u, v)$ are convergent series without free and linear terms) we know from the celebrated Poincaré–Lyapunov theorem that system (1) has a formal Lyapunov first integral of the form

$$\Phi(u, v) = u^2 + v^2 + \sum_{l+j=3}^{\infty} \phi_{l,j} u^l v^j$$

if and only if the origin is a centre on the real plane u, v and then the integral is analytical, and according to the Vorob'ev theorem [1,36] *the centre is isochronous if and only if system (1) is linearizable (in this case the formal transformation is also necessarily an analytical one).*

Using the complex variables $x = u + \mathrm{i}v$ we can write system (1) as a single equation:

$$\dot{x} = \mathrm{i}x + X(x, \bar{x}) \tag{2}$$

where $X(x, \bar{x}) = \sum_{k\geqslant 2} X_k(x, \bar{x})$ is an analytic function and $X_k(x, \bar{x})$ homogeneous polynomials of degree k. Due to the specific linear part this system (equation) has either a centre or a focus at the origin in the real plane $\{(u, v) \mid x = u + \mathrm{i}v\}$, while the saddles and nodes are excluded. For systems of the form (1) (equivalently, equation (2)) the notions of centre and isochronicity have a simple geometric meaning. Namely, the origin of system (1) is a centre if all trajectories in its neighbourhood are closed and it is an isochronous centre if the period of oscillations is the same for all these trajectories. However, a better understanding of integrability and isochronicity phenomena can be obtained by considering not only the real system (1) but also its complex form (2) along with the more general complex system

$$\dot{x} = \mathrm{i}x + X(x, y) \qquad \dot{y} = -\mathrm{i}y + Y(x, y) \tag{3}$$

where $X(x, y) = \sum_{k+l=2}^{\infty} X_{kl}x^k y^l$, $Y(x, y) = \sum_{k+l=2}^{\infty} Y_{kl}x^k y^l$ are series convergent in a neighbourhood of the origin. This system is equivalent to equation (2) in the case $x = \overline{y}$, $X_{ij} = \overline{Y}_{ji}$ and, after the change of time $\mathrm{i}\mathrm{d}t = \mathrm{d}\tau$, we transform it to

$$\mathrm{d}x/\mathrm{d}\tau = x + \tilde{X}(x, y) \qquad \mathrm{d}y/\mathrm{d}\tau = -y + \tilde{Y}(x, y). \tag{4}$$

We say that system (3) is linearizable if there is an analytic change of coordinate in the neighbourhood of the origin, bringing the system into the linear one. A nice geometric characterization of linearizable complex systems of the form (3) was obtained by Christopher and Rousseau.

Proposition 1 ([7]). *System (3) is linearizable if and only if there exists a neighbourhood of* $x = y = 0$ *such that every trajectory inside that neighbourhood is periodic.*

There are different algorithms to find the necessary conditions for linearizability; however, the most natural are the following two. One of them is a transformation of system (3) to the normal form

$$\dot{x}_1 = x_1\left(\mathrm{i} + \sum_{j=2}^{\infty} c_j(x_1y_1)^j\right) \qquad \dot{y}_1 = y_1\left(-\mathrm{i} + \sum_{j=2}^{\infty} d_j(x_1y_1)^j\right) \tag{5}$$

by means of the change of coordinate

$$x = x_1 + \sum_{k+j\geqslant 2} h_{kj}^{(1)} x_1^k y_1^j \qquad y = y_1 + \sum_{k+j\geqslant 2} h_{kj}^{(2)} x_1^k y_1^j. \tag{6}$$

Then the conditions of linearizability are the conditions

$$c_k = d_k = 0 \tag{7}$$

for all $k \geqslant 2$.

However, the calculations are much more efficient if instead of (6) we look for the transformation inverse to (6), namely,

$$x_1 = x + \sum_{k+j\geqslant 2} H_{kj}^{(1)} x^k y^j = \hat{H}^{(1)}(x, y) \qquad y_1 = y + \sum_{k+j\geqslant 2} H_{kj}^{(2)} x^k y^j = \hat{H}^{(2)}(x, y)$$

which brings system (3) to the linear system $\dot{x}_1 = \mathrm{i}x_1$, $\dot{y}_1 = -\mathrm{i}y_1$. Then the functions $\hat{H}^{(1)}$, $\hat{H}^{(2)}$ should satisfy the equations

$$\mathrm{i}\hat{H}^{(1)}(x, y) = \frac{\partial \hat{H}^{(1)}}{\partial x}\dot{x} + \frac{\partial \hat{H}^{(1)}}{\partial y}\dot{y} \qquad -\mathrm{i}\hat{H}^{(2)}(x, y) = \frac{\partial \hat{H}^{(2)}}{\partial x}\dot{x} + \frac{\partial \hat{H}^{(2)}}{\partial y}\dot{y}. \tag{8}$$

Equating coefficients of monomials $x^k y^j$ in these identities one can determine uniquely the coefficients $H_{kj}^{(1)}$, $H_{jk}^{(2)}$, when $j-k \neq 1$. When $j-k=1$ we meet the compatibility conditions (see section 2 for more details)

$$0 \cdot H_{k,k+1}^{(1)} = f_k^{(1)}(X, Y) \qquad 0 \cdot H_{k+1,k}^{(2)} = f_k^{(2)}(X, Y)$$

where $f_k^{(1)}(X, Y)$, $f_k^{(2)}(X, Y)$ are polynomials of the coefficients X_{lj}, Y_{lj} such that $l + j \leqslant 2k$. We call these polynomials the *linearizability (isochronicity)* quantities. Therefore the system is linearizable if and only if the infinite series of the conditions

$$f_1^{(1)}(X, Y) = f_1^{(2)}(X, Y) = \cdots = f_k^{(1)}(X, Y) = f_k^{(2)}(X, Y) = \cdots = 0 \tag{9}$$

is satisfied. We present a further development of this approach in section 3.

However, what do the linearizability conditions for system (3) have to do with the linearizability of the real system (1)? First of all the conditions (9) are the conditions of linearizability of system (3) which contains as a particular case equation (2) and, therefore, system (1). Hence, due to the Vorob'ev theorem [1, 36] equation (2) (= system (1)) has an isochronous centre at the origin if and only if

$$f_1^{(1)}(X, \bar{X}) = f_1^{(2)}(X, \bar{X}) = \cdots = f_k^{(1)}(X, \bar{X}) = f_k^{(2)}(X, \bar{X}) = \cdots = 0.$$

To see another connection we make, following [8], in equation (2) the substitution $r^2 = x\bar{x}$, $\theta = \arctan(\mathrm{Im}\,(x)/\mathrm{Re}\,(x))$. Then we get

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mathrm{i}r\frac{\dot{x}\bar{x} + x\dot{\bar{x}}}{\dot{x}\bar{x} - x\dot{\bar{x}}} = \frac{(X\bar{x} + \bar{X}x)/2r}{1 + (X\bar{x} - \bar{X}x)/(2\mathrm{i}r^2)} = \frac{\sum_{k\geqslant 2} r^k \mathrm{Re}\,(S_k(\theta))}{1 + \sum_{k\geqslant 1} r^k \mathrm{Im}\,(S_{k+1}(\theta))} \tag{10}$$

where $S_k(\theta) = \mathrm{e}^{-\mathrm{i}\theta} X_k(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})$. Denote by $r(\theta, \rho)$ the solution of the initial problem $r(0) = \rho$ for equation (10),

$$r(\theta, \rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \cdots \qquad \text{with} \quad u_k(\theta) = 0 \quad \text{for} \quad k \geqslant 2.$$

Let $P(\rho) = r(2\pi, \rho)$ be the return map defined on the $\theta = 0$ axis. The values $u_k(2\pi)$ (which generally speaking are polynomials in the parameters of system (2)) determine the behaviour of solutions (10) near the origin. Namely, if the first nonzero value $u_{2k+1}(2\pi)$ (it is well known that the first nonzero coefficient has a necessary odd subscript) is negative, then the system has a stable focus at the origin, if it is positive then the focus is unstable, and if all $u_k(2\pi)$ vanish then the origin is a centre. The quantity defined as $g_{2m+1} = u_{2m+1}(2\pi)$ is called the mth *Lyapunov quantity*.

Let now suppose that the origin is a centre for system (2), that is the functions $u_k(\theta)$ are periodic for all $k \geqslant 2$. The *period function*, $T(\rho)$, at ρ is defined as the time spent by the closed orbit $r(\theta, \rho)$ to turn once around the origin. A centre is isochronous if $T(\rho)$ is constant. We have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\mathrm{i}}{2r^2}(\dot{x}\bar{x} + x\dot{\bar{x}}) = 1 + \sum_{k\geqslant 2} r^{k-1} \mathrm{Im}\,(S_k(\theta)).$$

Hence, near the origin

$$T(\rho) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \sum_{k\geqslant 2} r(\theta, \rho)^{k-1} \mathrm{Im}\,(S_k(\theta))} = 2\pi + \sum_{k\geqslant 1} \int_0^{2\pi} \gamma_k(\theta)\,\mathrm{d}\theta\,\rho^k.$$

Denoting $t_k(\theta) = \int_0^\theta \gamma_k(s)\,\mathrm{d}s$ we get $T(\rho) = 2\pi + \sum_{k\geqslant 1} t_k(2\pi)\rho^k$. Obviously, the centre is isochronous if and only if $t_k(2\pi) = 0$ for $k \geqslant 1$. It is known [5, 8] that the first k with $t_k(2\pi) \neq 0$ is an even number. We call $T_{2m} = t_{2m}(2\pi)$ the mth *period quantity*. The function $T(\rho)$ and the values T_{2k} play a crucial role in the investigation of the problem of bifurcations of critical periods [5].

Note that neither linearizability quantities $f_k^{(1)}, f_k^{(2)}$ nor periodic constants T_{2k} are determined uniquely (the same is true also for focus and Lyapunov quantities). The uncertainty comes from integration constants for the T_{2k} values and from the possibility of choosing the coefficients $H_{k+1,k}^{(1)}, H_{k,k+1}^{(2)}$ arbitrarily, when we compute $f_k^{(1)}, f_k^{(2)}$. But the varieties defined by periodic and linearizability quantities in the space of parameters should not depend on the way they are calculated. However, it is much easier to handle the varieties in $\mathbb{C}^n$ than in $\mathbb{R}^n$. This (along with the computational efficiency of calculations of the linearizability quantities $f_k^{(1)}, f_k^{(2)}$) is the main reason why we prefer to work with the more general system (3) (or, in fact, (4)) rather than with the real system (1). Then, if we have the linearizability conditions for system (3) but we are interested in linearizability conditions for system (1), we get conditions after substituting Y_{kj} by $\bar{X}_{jk}$.

We conjecture that the following formula is valid:

$$T_{2k} \equiv f_k^{(1)}(X, \bar{X}) + f_k^{(2)}(X, \bar{X}) \mod J$$

where

$$J = \langle I + \langle f_1^{(1)}(X, \bar{X}) + f_1^{(2)}(X, \bar{X}), \ldots, f_{k-1}^{(1)}(X, \bar{X}) + f_{k-1}^{(2)}(X, \bar{X})\rangle\rangle$$

where I is the ideal generated by Lyapunov quantities (or defined below focus quantities), and $f_m^{(1)}, f_m^{(2)}$ are chosen such that $f_m^{(1)} = i_{mm}, f_m^{(2)} = j_{mm}$ with i_{mm}, j_{mm} defined in section 2. At present we cannot prove this formula but we have checked by using the expressions for T_{2k} given in [13] and our algorithm for computing focus and linearizability quantities obtained in section 3 (see also the appendix) that it is true for the case of equation (2) with $X(x, \bar{x})$ being a homogeneous polynomial of second degree. We remind the reader that an ideal of a ring is a subring closed under multiplication by any element of the ring.

Important results on the centre and isochronicity problems were obtained in the 1960s and 1970s and presently the problem is once again attracting considerable interest (see, e.g., [1, 3, 12, 18, 23, 33–39] and references therein). In recent years the use of computer algebra led to remarkable progress in the investigation of the centre and linearizability problems for polynomial dynamical systems. The study demands computing focus and linearizability quantities which are polynomials of the parameters of the polynomial system. However, the expressions are typically very large and in one of the directions of research considerable efforts have been devoted to developing methods and program packages for computing focus and linearizability (isochronicity) quantities for planar ODE systems (see, e.g. [4, 6, 13, 21, 22, 31] and references therein).

We also developed such algorithms: an algorithm for the computation of the focus quantities of polynomial systems of the form (11) was announced in [26] and one for linearizability quantities in [27]. In this paper we describe them in more detail. In our opinion our algorithms are the simplest and the most efficient because the calculation is reduced to just the summation and multiplication of rational numbers. The other important feature of our algorithms is their similarity: we use almost the same formula for computing both focus and linearizability quantities. In the appendix we present a Mathematica code for computing focus and linearizability quantities of a cubic system of differential equations based on our algorithm.

Using the algorithm we have computed up to 14 first linearizability quantities for 12 eight-parametric subfamilies of the cubic systems (the subfamilies consist of the systems with one quadratic term and three cubic terms per equation) and then we have resolved the linearizability problem for ten of these systems.

In the last section we demonstrate that there is a kind of duality between the phase space of a polynomial system and the space of its coefficients and between the Lie derivative along the polynomial vector field and a first-order linear partial differential operator.

As we have mentioned above, in fact, to solve the linearizability problem for a polynomial system means to verify whether the normal form of the system is linear. Therefore, developing methods for the investigation of the linearizability (isochronicity) problem we provide efficient algorithms for the transformation a system of differential equations to the linear normal form. To apply the ideas of this paper in order to improve the efficiency of transformations to nonlinear Birkhoff–Gustavson normal forms (in particular, in the spirit of [24]) is certainly one of our important ongoing projects.

2. Preliminaries

In this paper we will restrict our consideration to polynomial families of system (3). However, many expressions will look simpler if we apply the change of time $t \mapsto -\mathrm{i}t$, which transforms (3) into a system of the form (4). So, we will consider the polynomial system of the form (4)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \sum_{(p,q)\in S} a_{pq}x^{p+1}y^q = P(x,y) \\ -\frac{\mathrm{d}y}{\mathrm{d}t} &= y - \sum_{(p,q)\in S} b_{qp}x^q y^{p+1} = -Q(x,y) \end{aligned} \tag{11}$$

where x, y, a_{pq}, b_{qp} are complex variables, $S = \{(m,k)|m+k \geqslant 1\}$ is a subset of $\{-1\cup\mathbb{N}\}\times\mathbb{N}$, $\mathbb{N}$ is the set of non-negative integers. Let l be the number of elements in the set S. We denote by $E(a,b)(=\mathbb{C}^{2l})$ the parameter space of (11), and by $\mathbb{C}[a,b]$ ($\mathbb{Q}[a,b]$) the polynomial ring in the variables a_{pq}, b_{qp} over the field $\mathbb{C}$ (over $\mathbb{Q}$).

For system (11) one can always find a Lyapunov function

$$\Psi(x,y) = xy + \sum_{l+j\geqslant 3} v_{l-1,j-1}x^l y^j \tag{12}$$

such that

$$\mathrm{D}(\Psi) \stackrel{\text{def}}{=} \frac{\partial\Psi}{\partial x}P(x,y) + \frac{\partial\Psi}{\partial y}Q(x,y) = g_{11}(xy)^2 + g_{22}(xy)^3 + g_{33}(xy)^6 + \cdots. \tag{13}$$

Definition 1. *(1) The origin of system (11) with fixed coefficients $(a^*,b^*) \in E(a,b)$ is called a centre if there is a formal power series of the form (12)*

$$\Psi(x,y) = xy + \sum_{\substack{l+j=3\\ l,j\geqslant 0}}^{\infty} v_{l,j}(a^*,b^*)x^l y^j$$

such that

$$\mathrm{D}(\Psi) \equiv 0 \tag{14}$$

(implying $g_{kk} \equiv 0$ for all k, i.e. $\Psi(x,y)$ is a first integral of system (11)).

(2) The origin of system (11) with fixed coefficients $(a^*,b^*) \in E(a,b)$ is called a linearizable (or isochronical) centre if there is a formal change of coordinates

$$z_1 = x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a^*,b^*)x^m y^j \qquad z_2 = y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a^*,b^*)x^m y^j \tag{15}$$

which transforms (11) to the linear system

$$\dot{z}_1 = z_1 \qquad \dot{z}_2 = -z_2. \tag{16}$$

Remark. As it shown in [38, p 845] if $x = \bar{y}$ (this is the case of real system (2)) then $g_{2k+1} = \text{const} \cdot g_{kk}$ for a suitable choice of g_{2k+1} and g_{kk}.

Taking derivatives with respect to t in both parts of each equality of (15), we get

$$\dot{z}_1 = \dot{x} + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(mx^{m-1}y^j\dot{x} + jx^m y^{j-1}\dot{y}) \tag{17}$$

$$\dot{z}_2 = \dot{y} + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(mx^{m-1}y^j\dot{x} + jx^m y^{j-1}\dot{y}). \tag{18}$$

Equating coefficients of the terms $x^{q_1+1}y^{q_2}$, $x^{q_1}y^{q_2+1}$ in equations (17) and (18), correspondingly (instead of $\dot{z}_1, \dot{z}_2$, we substitute z_1 and $-z_2$, correspondingly, given by (15)) we obtain the recurrence formulae

$$(q_1 - q_2)u^{(1)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} [(s_1+1)u^{(1)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - s_2u^{(1)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}] \tag{19}$$

$$(q_1 - q_2)u^{(2)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} [s_1u^{(2)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)u^{(2)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}] \tag{20}$$

where $q_1, q_2 \geqslant -1, q_1 + q_2 \geqslant 0, u^{(1)}_{1,-1} = u^{(1)}_{-1,1} = 0, u^{(2)}_{00} = u^{(2)}_{00} = 1$, and we put $a_{qm} = b_{mq} = 0$, if $(q, m) \notin S$.

Thus we see that the coefficients $u^{(1)}_{q_1q_2}, u^{(2)}_{q_1q_2}$ of the transformation (15) can be computed step by step using formulae (19), (20). In the case $q_1 = q_2 = q$ the coefficients $u^{(1)}_{qq}, u^{(2)}_{qq}$ can be chosen arbitrarily (we set $u^{(1)}_{qq} = u^{(2)}_{qq} = 0$), but the system has a linearizable centre only if the quantities on the right-hand side of (19), (20) are equal to zero for all $q \in \mathbb{N}$. In the case $q_1 = q_2 = q$ we denote the polynomials in the right-hand side of (19) by i_{qq} and in the right-hand side of (20) by $-j_{qq}$.

We call the polynomial $g_{qq} \in \mathbb{Q}[a, b]$ in the right-hand side of (13) *qth focus quantities* and i_{qq}, j_{qq} the *qth linearizability (isochronicity) quantities* (we used the notation $f^{(1)}_q, -f^{(2)}_q$ for them in the previous section).

Definition 2. *The ideal in the polynomial ring $k[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_s$, denoted $\langle f_1, \ldots, f_s\rangle$, is the set*

$$\langle f_1, \ldots, f_s\rangle = \left\{ \sum_{i=1}^{s} u_i f_i \mid u_i \in k[x_1, \ldots, x_n],\ i = 1, \ldots, s \right\}.$$

The maximal set $V \subset E(a, b)$, where system (11) has a centre, is the set where all polynomials $g_{ii}, i = 1, 2, \ldots$ vanish, that is, V is the *variety* (the zero set) of the ideal generated by the focus quantities g_{ii}. Similarly, system (11) has a linearizable centre in the origin if and only if $i_{kk}(a^*, b^*) = j_{kk}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$.

Denote by $\boldsymbol{V}(I)$ the variety of the ideal I.

Definition 3. *The set*

$$V_C = \boldsymbol{V}(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots\rangle)$$

is called the centre variety of system (11).

So, for every point in V_C the corresponding system has a centre at the origin in the sense that there is a first integral of the form (12). However, if $(a, b) \in V_C$ and $a_{pq} = \bar{b}_{qp}$ for all $(p, q) \in S$, then such a point corresponds to a real system of the form (2), which then has a topological centre at the origin in the plane $x = u + \mathrm{i}v$. (For a geometrical interpretation of the centre of the complex system (11) see, e.g., [39].)

Definition 4. *The set*

$$V_{\mathcal{I}} = \boldsymbol{V}(\langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots, i_{kk}, j_{kk}, \ldots\rangle)$$

is called the linearizability (isochronicity) variety of system (11).

As we see we use the notions centre and isochronicity as synonyms of local integrability and linearizability, correspondingly.

3. The calculation of focus and linearizability quantities

In this section we present an efficient algorithm for computing focus and linearizability quantities of systems of the form (11). The algorithm is a further development of the methods presented in [21, 31].

We assume that $S = \{\bar{\imath}_1, \ldots, \bar{\imath}_l\}$, where $\bar{\imath}_s = (p_s, q_s)$ is the ordered set of the indices of the coefficients of the first equation of system (11) and consider the map $L : \mathbb{N}^{2l} \to \mathbb{N}^2$ (recall that l is the number of elements in the set S), defined by

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \nu_1\bar{\imath}_1 + \nu_2\bar{\imath}_2 + \cdots + \nu_{l-1}\bar{\imath}_{l-1} + \nu_l\bar{\imath}_l + \nu_{l+1}\bar{\jmath}_l + \nu_{l+2}\bar{\jmath}_{l-1} + \cdots + \nu_{2l-1}\bar{\jmath}_2 + \nu_{2l}\bar{\jmath}_1 \tag{21}$$

where $\bar{\jmath}_s$ corresponds to $\bar{\imath}_s$, such that if $\bar{\jmath}_s = \binom{p_s}{q_s}$, then $\bar{\imath}_s = \binom{q_s}{p_s}$.

Denote by M the monoid (= semigroup) of all solutions of the equation $L(\nu) = (k, k)^\top$, where k runs through the whole set $\{0, 1, 2, \ldots\}$ and by $\Omega(S)$ the monoid generated by the set S, so that $\Omega(S)$ contains all sums of elements of S. Let $\mathbb{Q}[M] \subset \mathbb{Q}[a_{\bar{\imath}_1}, \ldots, a_{\bar{\imath}_l}, b_{\bar{\jmath}_l}, \ldots, b_{\bar{\jmath}_1}] \stackrel{\text{def}}{=} \mathbb{Q}[a, b]$ be the monoid ring of the monoid M over $\mathbb{Q}$ and $\mathbb{Q}[\Omega(S)] \subset \mathbb{Q}[a, b]$ be the monoid ring of the monoid $\Omega(S)$.

Denote by $[\nu]$ the monomial

$$[\nu] = a_{\bar{\imath}_1}^{\nu_1} a_{\bar{\imath}_2}^{\nu_2} \ldots a_{\bar{\imath}_l}^{\nu_l} b_{\bar{\jmath}_l}^{\nu_{l+1}} b_{\bar{\jmath}_{l-1}}^{\nu_{l+2}} \ldots b_{\bar{\jmath}_1}^{\nu_{2l}}$$

and by $\overline{\nu}$ the involution of the vector ν

$$\overline{\nu} = (\nu_{2l}, \nu_{2l-1}, \ldots, \nu_2, \nu_1). \tag{22}$$

Definition 5. *Assume* $\binom{m}{n} \in \Omega(S)$. *A polynomial* $g \in \mathbb{Q}[\Omega(S)]$, $g = \sum_{\nu\in\operatorname{supp}(g)} a_{(\nu)}[\nu]$ *is called a* (m, n)*-polynomial if for every* $\nu \in \operatorname{supp}(g)$ *the condition* $L(\nu) = \binom{m}{n}$ *holds.*

It is easily seen that we can choose the coefficients $u_{kn}^{(1)}, u_{kn}^{(2)}$ such that

$$i_{qq} = -\bar{j}_{qq} \tag{23}$$

where by $\bar{j}_{qq}$ we denote the polynomial obtained as the result of the action the involution (22) on j_{qq}, i.e. as a result of replacing every monomial $[\nu]$ of j_{qq} by the monomial $[\bar{\nu}]$. In particular, this is the case when our assumption $u_{qq}^{(2)}, u_{qq}^{(1)} \equiv 0$ holds.

Theorem 1. *(1) There exists a formal series* $\Psi(x, y)$ *of the form (12) and polynomials* $g_{11}, g_{22}, \ldots$ *such that (13) holds and* $v_{ii} \equiv 0\ \forall i \geqslant 1$, $v_{kn} \in \mathbb{Q}[\Omega(S)]$ *and* v_{kn} *are* (k, n)*-polynomials,* $g_{ii} \in \mathbb{Q}[M]$ *and* g_{ii} *are* (i, i)*-polynomials for all* $(k, n) : k + n \geqslant 0,\ k, n \geqslant -1, i \geqslant 1$.

(2) The coefficients $u_{kn}^{(1)}, u_{kn}^{(2)}$ *of the transformation (15) are* (k, n)*-polynomials for all* $(k, n) : k + n \geqslant 0,\ k, n \geqslant -1$*; the linearizability quantities* i_{kk}, j_{kk} *belong to* $\mathbb{Q}[M]$ *for all* $k \geqslant 1$ *and are* (k, k)*-polynomials.*

Proof. (1) By equating the coefficients of the term $x^p y^q$ in identity (13) and putting $p-1=k_1, q-1=k_2$ we obtain the recurrence formula

$$(k_1-k_2)v_{k_1k_2} = \sum_{s_1+s_2=0}^{k_1+k_2-1} [(s_1+1)v_{s_1s_2}a_{k_1-s_1,k_2-s_2} - (s_1+1)v_{s_2s_1}b_{k_1-s_2,k_2-s_1}] \tag{24}$$

where $k_1, k_2 \geqslant -1$, $k_1+k_2 \geqslant 1$, $v_{00}=1$, $v_{1,-1}=v_{-1,1}=0$ and we set $a_{km}=b_{mk}=0$, if $(k,m) \notin S$.

We prove that $v_{k_1k_2} \in \mathbb{Q}[\Omega(S)]$ and $v_{k_1k_2}$ is a (k_1,k_2)-polynomial by induction on $m=k_1+k_2$. For $m=0$ the proposition holds. We assume that the proposition is true for all $m \leqslant m_0$. Let $k_1+k_2=m_0+1$, $k_1 \neq k_2$. Consider the product

$$v_{s_1s_2}a_{k_1-s_1,k_2-s_2} = \left(\sum_{\nu\in \mathrm{supp}\,(v_{s_1s_2})} \alpha_{(\nu)}[\nu]\right)[\mu]$$

where $\mu=(0,\ldots,0,1,0,\ldots,0)$ such that 1 stands at the place corresponding to vector (k_1-s_1, k_2-s_2) in the ordered set S. $\mu \in \Omega(S)$ and due to the induction assuming $\nu \in \Omega(S)$. Therefore, $\nu+\mu \in \Omega(M)$ and hence

$$v_{s_1s_2}a_{k_1-s_1,k_2-s_2},\ v_{s_2s_1}b_{k_1-s_2,k_2-s_1} \in \mathbb{Q}[\Omega(S)].$$

Taking into account that $L(\mu)=\binom{k_1-s_1}{k_2-s_2}$ and thanks to the induction hypothesis we conclude that $L(\nu+\mu)=\binom{k_1}{k_2}$. Therefore $v_{k_1k_2}$ is a (k_1,k_2)-polynomial.

If $k_1=k_2=k$ we choose $v_{kk}\equiv 0$ and g_{kk} equal to the right-hand side of (24). As above from (24) we conclude that g_{kk} is (k,k)-polynomial.

The second statement is proven similarly. □

Corollary 1. *If* $(m,n) \notin \Omega(S)$, *then* $v_{mn}, u^{(1)}_{mn}, u^{(2)}_{mn} \equiv 0$.

Consider the formal series

$$V^{(\alpha,\beta)} = \sum V^{(\alpha,\beta)}_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a^{\nu_1}_{\bar{\imath}_1} a^{\nu_2}_{\bar{\imath}_2}\ldots a^{\nu_l}_{\bar{\imath}_l} b^{\nu_{l+1}}_{\bar{\jmath}_l} b^{\nu_{l+2}}_{\bar{\jmath}_{l-1}}\ldots b^{\nu_{2l}}_{\bar{\jmath}_1} \tag{25}$$

where α, β can have the values 0 or 1 and $V^{(\alpha,\beta)}_{(\nu_1,\ldots,\nu_{2l})}$ are determined by the following recurrence formula:

$$V^{(\alpha,\beta)}_{(\nu_1,\nu_2,\ldots,\nu_{2l})} = \frac{1}{L^1(\nu)-L^2(\nu)}\left(\sum_{i=1}^{l} V^{(\alpha,\beta)}_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^1(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+\alpha)\right.$$
$$\left. - \sum_{i=l+1}^{2l} V^{(\alpha,\beta)}_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^2(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+\beta)\right) \tag{26}$$

if $L^1(\nu)\neq L^2(\nu)$, $V^{(\alpha,\beta)}_{(\nu_1,\ldots,\nu_{2l})}=0$, if $L^1(\nu)=L^2(\nu)$; $V^{(\alpha,\beta)}_{(0,\ldots,0)}=1$ and we put $V^{(\alpha,\beta)}_{(\nu_1,\ldots,\nu_{2l})}=0$ for all $\nu=(\nu_1,\ldots,\nu_{2l})$, such that there exists $i : \nu_i < 0$.

Theorem 2. *(1) The coefficient of* $[\nu]$ *in the polynomial* v_{kn} *is equal to* $V^{(1,1)}_{(\nu_1,\nu_2,\ldots,\nu_{2l})}$ *computed according to (26) with* $\alpha=\beta=1$.

(2) The ith *focus quantity of system (11) is*

$$g_{ii} = \sum_{\nu:L(\nu)=\binom{i}{i}} g_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a^{\nu_1}_{\bar{\imath}_1} a^{\nu_2}_{\bar{\imath}_2}\ldots a^{\nu_l}_{\bar{\imath}_l} b^{\nu_{l+1}}_{\bar{\jmath}_l} b^{\nu_{l+2}}_{\bar{\jmath}_{l-1}}\ldots b^{\nu_{2l}}_{\bar{\jmath}_1} \tag{27}$$

where

$$g_{(\nu_1,\nu_2,\ldots,\nu_{2l})} = \sum_{i=1}^{l} V^{(1,1)}_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^1(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+1)$$
$$- \sum_{i=l+1}^{2l} V^{(1,1)}_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^2(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+1) \tag{28}$$

and $V^{(1,1)}_{(\nu)}$ *are defined by (26) with* $\alpha = \beta = 1$.

(3) The following property occurs:

$$V^{(1,1)}_{(\nu)} = V^{(1,1)}_{(\bar\nu)} \qquad g_{(\nu)} = -g_{(\bar\nu)} \qquad \text{if} \quad \nu \neq \bar\nu \qquad V^{(1,1)}_{(\nu)} = g_{(\nu)} = 0 \qquad \text{if} \quad \nu = \bar\nu. \tag{29}$$

(4) The coefficient of $[\nu]$ *in the polynomial* $u^{(1)}_{kn}$ *and the coefficient of* $[\bar\nu]$ *in the polynomial* $u^{(2)}_{nk}$ *are equal to* $V^{(1,0)}_{(\nu_1,\nu_2,\ldots,\nu_{2l})}$*, where* $V^{(1,0)}_{(\nu_1,\nu_2,\ldots,\nu_{2l})}$ *is computed according to (26) with* $\alpha = 1$, $\beta = 0$.

(5) The linearizability quantity i_{kk} *of system (11) is equal to*

$$i_{kk} = \sum_{\nu:L(\nu)=(k,k)^\top} i_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a^{\nu_1}_{\bar\imath_1} a^{\nu_2}_{\bar\imath_2} \ldots a^{\nu_l}_{\bar\imath_l} b^{\nu_{l+1}}_{\bar\jmath_l} b^{\nu_{l+2}}_{\bar\jmath_{l-1}} \ldots b^{\nu_{2l}}_{\bar\jmath_1}$$

where

$$\begin{aligned} i_{(\nu_1,\nu_2,\ldots,\nu_{2l})} = &\sum_{k=1}^{l} V^{(1,0)}_{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2l})}(L^1(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2l})+1) \\ &- \sum_{j=l+1}^{2l} V^{(1,0)}_{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2l})} L^2(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2l}) \end{aligned}$$

and $V^{(1,0)}_{(\nu)}$ *are given by (26) with* $\alpha = 1$, $\beta = 0$.

Proof. (1) According to theorem 1 coefficients v_{kn} of the function Ψ uniquely define the formal series (25) and, vice versa, every function of the form (25) uniquely defines the corresponding set of (k, n)-polynomials.

If ν is such that $L^1(\nu) \neq L^2(\nu)$ then by equating the coefficients of $[\nu]$ in the left- and the right-hand sides of (24) we obtain that formula (26) occurs. If $L^1(\nu) = L^2(\nu)$ then we set $V^{(\alpha,\beta)}_{(\nu)} = 0$ because we choose $v_{L^1(\nu),L^2(\nu)} \equiv 0$.

(2) The ith focus quantity is equal to the right-hand side of formula (24) in the case $i = k_1 = k_2$. From this fact and theorem 1 the proposition follows.

(3) Taking into account that $L^1(\nu) = L^2(\bar\nu)$, $L^2(\nu) = L^1(\bar\nu)$, from (26), (28) we get that formula (29) holds.

The statements (4) and (5) are proven similarly to (1) and (2), respectively. □

Note that in order to compute the linearizability quantities $-j_{kk}$ one can use formula (26) with $\alpha = 0$, $\beta = 1$, but in fact we immediately obtain these quantities from i_{kk} using formula (23).

Thus we see that the linearizability quantities can be computed using almost the same formulae which we obtained for the focus quantities. To compute the quantities one needs only to apply summation and multiplication rational number operations. The formulae are also very easily programmed. For example, using Mathematica one can write a code with the formulae practically in the same form (25), (26) as they are given in this paper (see the appendix).

Using statements (2) and (3) of theorem 2 we get the following important result.

Corollary 2. *The focus quantities have the form*

$$g_{ii} = \sum_{\nu:L(\nu)=\binom{i}{i}} g_{(\nu)}([\nu] - [\bar\nu]). \tag{30}$$

4. The linearizability conditions for some cubic systems

We consider the complex cubic system of the type (11)

$$\begin{aligned}\dot{x} &= x(1 - a_{10}x - a_{01}y - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3) \\ &= P(x, y) \\ \dot{y} &= -y(1 - b_{2,-1}x^2y^{-1} - b_{10}x - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2) \\ &= Q(x, y).\end{aligned} \tag{31}$$

In this section we obtain the necessary and sufficient conditions of linearizability of a centre for some eight-parametric subfamilies of system (31). Namely, we consider the systems where one of the coefficients a_{10}, a_{01}, a_{-12} differs from zero and one of the coefficients $a_{20}, a_{11}, a_{02}, a_{-13}$ is equal to zero. If $\{a^*_{ij}\}$ is such a four-parametric set, we keep in the second equation of system (31) the corresponding $\{b^*_{ji}\}$ parameters differing from zero and set the others equal to zero. Obviously, there are 12 such eight-parametric systems. We will encode the set of these systems using the parameters of the first equation equal to zero. For example, writing $a_{10} = a_{-12} = a_{-13} = 0$ denotes the system

$$\begin{aligned}\dot{x} &= x(1 - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2) \\ \dot{y} &= -y(1 - b_{10}x - b_{20}x^2 - b_{11}xy - b_{02}y^2).\end{aligned} \tag{32}$$

To solve the linearizability problem for these systems we computed the first linearizability quantities i_{kk}, j_{kk} up to $k = 7$. The polynomials are too long, so we do not present them here; however, one can easily check our calculations using Mathematica or any other computer algebra system (our Mathematica code for computing the linearizability quantities is presented in the appendix). Then using the computer algebra program Singular [17] (one can also use, e.g. CALI [15] or Macaulay [16]) we find the primary decomposition of the ideal $\langle i_{11}, j_{11}, \ldots, i_{77}, j_{77}\rangle$ and obtain the necessary conditions of linearizability. To prove that the obtained conditions are also the sufficient conditions for the centre to be an isochronical (linearizable) one we look for a Darboux linearization [7,23].

Definition 6. *We call a Darboux linearization of system (11) a change of variables*

$$z_1 = H_1(x, y) \qquad z_2 = H_2(x, y) \tag{33}$$

which transforms the system to the linear one, $\dot{z}_1 = z_1, \dot{z}_2 = -z_2$, and such that at least one of the functions H_1, H_2 is of the form

$$H = f_1^{\alpha_1} \cdots f_k^{\alpha_k} \tag{34}$$

α_j being complex numbers, where the $f_i(x, y)$ are invariant algebraic curves of system (11) defined by $f_i(x, y) = 0$, that is, polynomials satisfying the equation

$$\frac{\partial f_i}{\partial x} P + \frac{\partial f_i}{\partial y} Q = K_i f_i. \tag{35}$$

The polynomial $K_i(x, y)$ is called the cofactor of the invariant curve $f_i(x, y)$.

There are two commonly used possibilities to construct the first integral. The first one is if

$$\sum_{i=1}^{k} \alpha_i K_i = 0 \tag{36}$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a first integral of system (11). The second one is if the equality

$$\sum_{i=1}^{s} \beta_i K_i + P'_x + Q'_y = 0 \tag{37}$$

is satisfied (where again β_j are complex numbers), because then it yields the integrating factor $\mu = f_1^{\beta_1} \cdots f_s^{\beta_s}$ of the equation $Q(x, y)\,\mathrm{d}x - P(x, y)\,\mathrm{d}y = 0$.

Now we seek the linearizing transformation. It can be shown [23] that if

$$P(x, y)/x + \sum_{i=1}^{k} \alpha_i K_i = 1 \tag{38}$$

then after the substitution

$$z_1 = x f_1^{\alpha_1} \cdots f_k^{\alpha_k} \tag{39}$$

we get

$$\dot{z}_1 = z_1$$

and if

$$Q(x, y)/y + \sum_{i=1}^{k} \alpha_i K_i = -1 \tag{40}$$

then the second equation of system (11) is linearized by the change

$$z_2 = y f_1^{\alpha_1} \cdots f_k^{\alpha_k}. \tag{41}$$

On the other hand, if system (11) is such that only one of the conditions (38) and (40) is satisfied, let us say (40), but system (11) has a Lyapunov first integral $\Psi(x, y)$ of the form (12), then (11) is linearizable by the change

$$z_1 = \Psi(x, y)/H_2(x, y) \qquad z_2 = H_2(x, y) \tag{42}$$

and, correspondingly, if (38) is satisfied, then the linearizing transformation is given by

$$z_1 = H_1(x, y) \qquad z_2 = \Psi(x, y)/H_1(x, y) \tag{43}$$

as can be verified by a straightforward calculation [7].

Theorem 3. *Table 1 takes place.*

Proof. We say that the linearizability condition A is 'symmetric' to the linearizability condition B if after replacing any a_{ji}, b_{kl} in A by b_{ij}, a_{lk}, correspondingly, we get B. Obviously it is sufficient to consider only one of the 'symmetric' conditions.

(I)-(1) This case is a particular one of (II) (1) considered below and (2) is 'symmetric' to (1). Below we will mention only one of the 'symmetric' cases.

(II) In this case the system has the form

$$\dot{x} = x(1 - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{02}y^2 - a_{-13}x^{-1}y^3)$$
$$\dot{y} = -y(1 - b_{2,-1}x^2y^{-1} - b_{20}x^2 - b_{02}y^2 - b_{3,-1}x^3y^{-1}).$$

Here (4), (5), (7) and (9) are particular cases of the system with homogeneous cubic nonlinearities considered in [7]. The case (3) is the same as (IV)-(2) and the case (8) is identical to (IV)-(5) examined below. In the case (1) the so-called symmetry conditions [18, 20, 31] are satisfied, therefore we have a first integral (12) and the linearizing substitution

$$z_1 = \frac{F(x, y)\sqrt{1 - b_{02}y^2}}{y} \qquad z_2 = \frac{y}{\sqrt{1 - b_{02}y^2}}. \tag{44}$$

(III) When the conditions (1) are fulfilled the corresponding system is a particular case of (II)-(1) and the case (3) is considered in [7].

(IV) We present the proof for this case in section 5.

Table 1. The isochronicity varieties

Case	System	Linearizability (isochronicity) conditions
I	$a_{10} = a_{01} = a_{20} = 0$	(1) $b_{2,-1} = b_{20} = b_{3,-1} = b_{11} = a_{11} = 0$; (2) $a_{-13} = a_{02} = a_{-12} = b_{11} = a_{11} = 0$;
II	$a_{10} = a_{01} = a_{11} = 0$	(1) $b_{2,-1} = b_{20} = b_{3,-1} = a_{20} = 0$; (2) $b_{02} = a_{-13} = a_{02} = a_{-12} = 0$: (3) $b_{2,-1} = b_{3,-1} = a_{-13} = a_{02} + b_{02} = a_{20} + b_{20} = 0$: (4) $b_{2,-1} = 112b_{20}^3 + 27b_{3,-1}^2 b_{02} = 49a_{-13}b_{20}^2 - 9b_{3,-1}b_{02}^2 = 21a_{-13}b_{3,-1} + 16b_{20}b_{02} = 343a_{-13}^2 b_{20} + 48b_{02}^3 = 7a_{02} + 3b_{02} = 3a_{20} + 7b_{20} = a_{-12} = 0$: (5) $b_{2,-1} = b_{02} = b_{3,-1} = a_{02} = a_{20} + 3b_{20} = a_{-12} = 0$: (6) $b_{3,-1} = a_{-13} = a_{02} + b_{02} = a_{20} + b_{20} = a_{-12} = 0$: (7) $b_{2,-1} = b_{20} = a_{-13} = 3a_{02} + b_{02} = a_{20} = a_{-12} = 0$; (8) $b_{2,-1} = b_{02} = b_{3,-1} = a_{-13} = a_{02} = a_{20} + 2b_{20} = 0$; (9) $b_{2,-1} = b_{20} = b_{3,-1} = a_{-13} = a_{02} = a_{-12} = 0$; (10) $b_{20} = b_{3,-1} = a_{-13} = 2a_{02} + b_{02} = a_{20} = a_{-12}$;
III	$a_{10} = a_{01} = a_{02} = 0$	(1) $b_{2,-1} = b_{3,-1} = a_{20} = b_{11} = a_{11} = 0$; (2) $b_{02} = a_{-13} = a_{-12} = b_{11} = a_{11} = 0$; (3) $b_{2,-1} = b_{3,-1} = a_{-13} = a_{-12} = b_{11} = a_{11} = 0$;
IV	$a_{10} = a_{01} = a_{-13} = 0$	(1) $b_{2,-1} = b_{20} = a_{20} = b_{11} = a_{11} = 0$; (2) $b_{2,-1} = a_{02} + b_{02} = a_{20} + b_{20} = b_{11} = a_{11} = 0$; (3) $b_{02} = a_{02} = a_{-12} = b_{11} = a_{11} = 0$; (4) $a_{02} + b_{02} = a_{20} + b_{20} = a_{-12} = b_{11} = a_{11} = 0$; (5) $b_{2,-1} = b_{02} = a_{02} = a_{20} + 2b_{20} = b_{11} = a_{11} = 0$; (6) $b_{2,-1} = b_{20} = a_{02} = a_{-12} = b_{11} = a_{11} = 0$; (7) $b_{20} = 2a_{02} + b_{02} = a_{20} = a_{-12} = b_{11} = a_{11} = 0$;
V	$a_{10} = a_{-12} = a_{11} = 0$	(1) $b_{10} = b_{20} = b_{3,-1} = a_{20} = 0$; (2) $b_{02} = a_{-13} = a_{02} = a_{01} = 0$; (3) $b_{10} = b_{20} = b_{3,-1} = a_{-13} = 2a_{01}^2 + a_{02} = 0$; (4) $b_{02} = 6b_{10}^2 + b_{20} = b_{3,-1} = a_{02} = a_{20} - 9b_{01}^2 = a_{01} = 0$; (5) $b_{10} = 112b_{20}^3 + 27b_{3,-1}^2 b_{02} = 49a_{-13}b_{20}^2 - 9b_{3,-1}b_{02}^2 = 21a_{-13}b_{3,-1} + 16b_{20}b_{02} = 343a_{-13}^2 b_{20} + 48b_{02}^3 = 7a_{02} + 3b_{02} = 3a_{20} + 7b_{20} = a_{01} = 0$; (6) $b_{10} = b_{02} = b_{3,-1} = a_{02} = a_{20} + 3b_{20} = a_{01} = 0$; (7) $2b_{10}^2 + b_{20} = b_{3,-1} = a_{-13} = a_{02} = a_{01} = 0$; (8) $b_{10} = b_{3,-1} = a_{-13} = a_{02} + b_{02} = a_{20} + b_{20} = a_{01} = 0$; (9) $b_{10} = b_{20} = a_{-13} = a_{02} + 6a_{01}^2 = a_{20} = 9a_{01}^2 - b_{02} = 0$; (10) $b_{10} = b_{20} = a_{-13} = 3a_{02} + b_{02} = a_{20} = a_{01} = 0$;
VI	$a_{10} = a_{-12} = a_{02} = 0$	(1) $b_{10} = b_{11} = b_{3,-1} = a_{20} = a_{11} = 0$; (2) $b_{02} = b_{11} = a_{-13} = a_{01} = a_{11} = 0$; (3) $a_{20} = b_{10}^2, b_{02} = a_{01}^2, b_{11} = a_{11} = -a_{01}b_{10}, a_{-13} = b_{3,-1} = 0$; (4) $b_{10} = b_{11} = b_{3,-1} = a_{-13} = a_{01} = a_{11} - b_{11} = 0$;
VII	$a_{10} = a_{-12} = a_{-13} = 0$	(1) $a_{02} = b_{20} = a_{20} - b_{10}^2 = a_{01}b_{10} + b_{11} = a_{01}b_{10} + a_{11} = a_{01}^2 - b_{02} = 0$; (2) $b_{02} = b_{11} = a_{11} = a_{02} = a_{01} = 0$; (3) $a_{02} = a_{01} = a_{11} = b_{11} = b_{20} + 2b_{10}^2 = 0$; (4) $b_{10} = b_{20} = b_{11} = a_{11} = a_{20} = 0$; (5) $b_{10} = b_{20} = b_{11} = a_{11} = a_{02} + 2a_{01}^2 = 0$; (6) $b_{10} = a_{01} = a_{11} = b_{11} = a_{02} + b_{02} = a_{20} + b_{20} = 0$;
VIII	$a_{01} = a_{-12} = a_{-13} = 0$	(1) $b_{02} = a_{02} = b_{11} = a_{11} = 0$; (2) $a_{02} + b_{02} = a_{20} + b_{20} = b_{11} = a_{11} = 0$; (3) $b_{20} = a_{02} = b_{11} = a_{11} = 0$; (4) $b_{20} = a_{20} = b_{11} = a_{11} = 0$;
IX	$a_{01} = a_{-12} = a_{02} = 0$	(1) $b_{3,-1} = a_{-13} = b_{11} = a_{11} = 0$; (2) $b_{02} = a_{-13} = b_{11} = a_{11} = 0$; (3) $a_{-13} + b_{02} = a_{20} + b_{3,-1} = b_{11} = a_{11} = 0$; (4) $b_{3,-1} = a_{20} = b_{11} = a_{11} = 0$;
X	$a_{01} = a_{-12} = a_{20} = 0$	(1) $b_{20} = b_{3,-1} = b_{11} = a_{11} = 0$; (2) $a_{-13} = a_{02} = b_{11} = a_{11}$

(V) Here cases (5), (6), (8) are the particular ones from [7]. In case (1) we again have the symmetry component of the centre variety [18, 20, 31] and, hence, there is a Lyapunov integral and the linearizing substitution is given by the formula (44).

When the conditions (3) hold the system has four invariant lines

$$l_1 = x \qquad l_2 = y \qquad l_3 = 1 - \sqrt{b_{02}}\,y \qquad l_4 = 1 + \sqrt{b_{02}}\,y$$

and, if $b_{02} \neq 0$, then there exists the integrating factor

$$\mu = \frac{1}{x^3\,y^3} {l_3}^{(2\,a_{01}^2 - a_{01}\,\sqrt{b_{02}})/b_{02}}\,{l_4}^{(2\,a_{01}^2 + a_{01}\,\sqrt{b_{02}})/b_{02}}.$$

The factor yields non-holomorphic first integral; however, then a first integral of the form (12) should also exist. Therefore, the system is linearized by the change (44). A linearizing change of variable there exists also when $b_{02} = 0$ because the components of the linearizability variety are closed sets in Zarizki topology.

We failed to find linearizing transformations in cases (4) and (9), so, in fact, in these cases the statements (V)-(4) and (V)-(9) are just hypotheses.

(VI) Case (1) is a subcase of (V)-(1) and (3), (4) are particular cases from [27].

(VII) This case was considered in [27]. Here, in case (1), using another approach, we found a new type of linearizing transformation previously unknown. We present the approach and the treatment of this case in section 5.

(VIII) In [28] we found for this system the following linearizing substitutions:

(1)

$$z_1 = x\,{s_1}^{b_{01}/2\,\sqrt{-4\,a_{02}+b_{01}^2} - a_{10}/2\,\sqrt{a_{10}^2 - 4\,b_{20}}}\,{s_3}^{-\frac{1}{2} + a_{10}/2\sqrt{a_{10}^2 - 4\,b_{20}}}\,{s_2}^{-\frac{1}{2} - b_{01}/2\sqrt{-4\,a_{02}+b_{01}^2}}$$

$$z_2 = y\,{s_1}^{-b_{01}/2\,\sqrt{-4\,a_{02}+b_{01}^2} + a_{10}/2\,\sqrt{a_{10}^2 - 4\,b_{20}}}\,{s_2}^{-\frac{1}{2} + b_{01}/2\,\sqrt{-4\,a_{02}+b_{01}^2}}\,{s_3}^{-\frac{1}{2} - a_{10}/2\,\sqrt{-4\,b_{20}+a_{10}^2}}$$

where

$$s_1 = 1 - \tfrac{1}{2}\left(a_{10} + \sqrt{a_{10}^2 - 4\,b_{20}}\right)x - \tfrac{1}{2}\left(b_{01} + \sqrt{-4\,a_{02} + b_{01}^2}\right)y$$

$$s_2 = 1 - \tfrac{1}{2}\left(a_{10} + \sqrt{a_{10}^2 - 4\,b_{20}}\right)x - \tfrac{1}{2}\left(b_{01} - \sqrt{-4\,a_{02} + b_{01}^2}\right)y$$

$$s_3 = 1 - \tfrac{1}{2}\left(a_{10} - \sqrt{a_{10}^2 - 4\,b_{20}}\right)x - \tfrac{1}{2}\left(b_{01} + \sqrt{-4\,a_{02} + b_{01}^2}\right)y$$

$$s_4 = 1 - \tfrac{1}{2}\left(a_{10} - \sqrt{a_{10}^2 - 4\,b_{20}}\right)x - \tfrac{1}{2}\left(b_{01} - \sqrt{-4\,a_{02} + b_{01}^2}\right)y.$$

(2)

$$z_1 = x\,{s_1}^{-\frac{1}{2} - a_{10}/2\sqrt{4\,a_{20}+{a_{10}}^2}}\,{s_2}^{-\frac{1}{2} + a_{10}/2\sqrt{{a_{10}}^2+4\,a_{20}}} \qquad z_2 = y\,{s_3}^{-\frac{1}{2} - b_{01}/2\,\sqrt{4\,b_{02}+{b_{01}}^2}}\,{s_4}^{-\frac{1}{2} + b_{01}/2\,\sqrt{4\,b_{02}+{b_{01}}^2}}.$$

with

$$s_1 = 1 - \tfrac{1}{2}\left(a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}\right)x \qquad s_2 = 1 - \tfrac{1}{2}\left(a_{10} - \sqrt{a_{10}^2 + 4\,a_{20}}\right)x$$

$$s_3 = 1 - \tfrac{1}{2}\left(b_{01} + \sqrt{4\,b_{02} + b_{01}^2}\right)y \qquad s_4 = 1 - \tfrac{1}{2}\left(b_{01} - \sqrt{4\,b_{02} + b_{01}^2}\right)y.$$

(3)

$$z_1 = \Psi(x, y)/H(x, y) \qquad z_2 = H(x, y)$$

where

$$H(x, y) = y\,{s_3}^{-\frac{1}{2} - b_{01}/2\,\sqrt{4\,b_{02}+b_{01}^2}}\,{s_4}^{-\frac{1}{2} + b_{01}/2\,\sqrt{4\,b_{02}+b_{01}^2}}$$

$s_3,\ s_4$ are given above and $\Psi(x, y)$ is the Lyapunov integral (12) of the system. Using theorem 1 from [26] it is easy to see that the function Ψ/H is analytic.

Taking into account that the components of the linearizability variety must be closed in Zariski topology we conclude that the corresponding systems are also linearizable when $a_{10}^2 = -4a_{20},\ b_{01}^2 = -4b_{02},\ a_{10}^2 = 4b_{20},\ b_{01}^2 = 4a_{02}$.

(IX) (1) In this case there are four invariant lines:

$$\begin{aligned} &l_1 = 1 - \tfrac{1}{2}\left(a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}\right)x \qquad && l_2 = 1 + \tfrac{1}{2}\left(-a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}\right)x \\ &l_3 = 1 - \tfrac{1}{2}\left(b_{01} + \sqrt{b_{01}^2 + 4\,b_{02}}\right)y \qquad && l_4 = 1 + \tfrac{1}{2}\left(-b_{01} + \sqrt{b_{01}^2 + 4\,b_{02}}\right)y \end{aligned} \tag{45}$$

and the linearization is defined by

$$z_1 = x l_1^{\alpha_1} l_2^{\alpha_2} \qquad z_2 = y l_3^{\alpha_3} l_4^{\alpha_4}$$

where

$$\begin{aligned} &\alpha_1 = -\frac{a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}}{2\sqrt{a_{10}^2 + 4\,a_{20}}} \qquad && \alpha_2 = -\frac{-a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}}{2\sqrt{a_{10}^2 + 4\,a_{20}}} \\ &\alpha_3 = -\frac{b_{01} + \sqrt{b_{01}^2 + 4\,b_{02}}}{2\sqrt{b_{01}^2 + 4\,b_{02}}} \qquad && \alpha_4 = \frac{b_{01} - \sqrt{b_{01}^2 + 4\,b_{02}}}{2\sqrt{b_{01}^2 + 4\,b_{02}}}. \end{aligned} \tag{46}$$

(2) To prove that the system

$$\dot{x} = x(1 - a_{10}x - a_{20}x^2) \qquad \dot{y} = -y(1 - b_{01}y - b_{3,-1}x^3y^{-1}) \tag{47}$$

has a centre in the origin we use the method developed in [11].

Expanding the equation of trajectories into the power series we get

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sum_{i=0}^{\infty} a_i x^i \tag{48}$$

where

$$a_{3k+1} = \frac{B^k}{\alpha(y)^{k+1}} \qquad a_{3k+2} = \frac{-a_{10}B^k}{\alpha(y)^{k+1}} \qquad a_{3k+3} = \frac{-a_{20}B^k}{\alpha(y)^{k+1}} \tag{49}$$

$k \geqslant 0$, $B = -b_{3,-1}$ and $\alpha(y) = y(b_{01}y - 1)$. We look for a first integral of system (47) of the form

$$H = \sum_{i=1}^{\infty} H_i(y)x^i. \tag{50}$$

Then the functions H_i satisfy the differential equations

$$\begin{aligned} &H_1' + a_1 H_1 = 0, \\ &H_2' + 2a_1 H_2 = -a_2 H_1, \\ &\vdots \\ &H_k' + k a_1 H_k = f_k, \\ &\vdots \end{aligned} \tag{51}$$

where

$$f_k = -(k-1)a_2 H_{k-1} - (k-2)a_3 H_{k-2} - \cdots - a_k H_1.$$

From the first equation we get

$$H_1 = \frac{y}{b_{01}y - 1} \tag{52}$$

and the next five yield

$$\begin{aligned} &H_2 = -\frac{a_{10}H_1^2}{y} \qquad H_3 = \frac{H_1^3 P_2(y)}{y^3} \qquad H_4 = \frac{H_1^4 P_2(y)}{y^4} \\ &H_5 = \frac{H_1^5 P_3(y)}{y^5} \qquad H_6 = \frac{H_1^6 P_3(y)}{y^6} \end{aligned} \tag{53}$$

where here and below we denote by P_i any polynomials of degree i.

We show that the coefficients H_m have the following general form for $m \geqslant 1$:

$$H_{6k+s}(y) = \begin{cases} \dfrac{H_1(y)^{6k+s} P_{5k+[\frac{s-1}{2}]}(y)}{y^{8k+s-1}} & \text{when} \quad s = 1, 2, 3 \\ \dfrac{H_1(y)^{6k+s} P_{5k+[\frac{s+1}{2}]}(y)}{y^{8k+s}} & \text{when} \quad s = 4, 5, 6 \end{cases} \tag{54}$$

where $[a]$ denotes the integer part of a.

We prove the statement by induction on k. According to (52), (53) for $k = 0$ it holds. Let us suppose that the formula is proven for $k < m$ and consider the case $k = m$. Note that

$$H_r(y) = H_1(y)^r \int^y f_r(u) H_1(u)^{-r} \, \mathrm{d}u \tag{55}$$

where $f_r = \sum_{i=1}^{r-1} (r - i) a_{i+1} H_{r-i}$, and

$$\int^y \frac{P_s(u)}{u^n} \, \mathrm{d}u = \frac{P_s(y)}{y^{n-1}} \tag{56}$$

when $n > s + 1$ (of course, the polynomials P_m in the right-hand side of (56) and the left-hand side are different, but for us only the degree is important, so we use the same notation P_r for any polynomial of degree r). Using (55) for $k = m$ we have

$$H_{6m+s}(y) = -H_1(y)^{6m+s} \int^y \left(\sum_{i=2}^{m} (6m + s + 1 - i) a_i(u) H_{6m+s+1-i}(u) \right) H_1(u)^{-6m-s} \, \mathrm{d}u.$$

Therefore to prove (54) it is sufficient to show that

$$\int^y a_{3k+l}(u) H_{6m+s+1-3k-l}(u) H_1(u)^{-6m-s} \, \mathrm{d}u = \begin{cases} \dfrac{P_{5m+[\frac{s-1}{2}]}(y)}{y^{8m+s-1}} & \text{when} \quad s = 1, 2, 3 \\ \dfrac{P_{5m+[\frac{s+1}{2}]}(y)}{y^{8m+s}} & \text{when} \quad s = 4, 5, 6. \end{cases} \tag{57}$$

for $l = 1, 2, 3$.

Let us consider the case $l = 3$. Then

$$\begin{aligned} &\int^y a_{3k+3}(u) H_{6m+s-3k-2}(u) H_1(u)^{-6m-s} \, \mathrm{d}u \\ &\qquad = C \int^y \frac{1}{u^{k+1}(b_{01}u - 1)^{k+1}} H_{6m+s-3k-2}(u) H_1(u)^{-6m-s} \, \mathrm{d}u \end{aligned} \tag{58}$$

where C is a constant. Assume that k is even. Then (49), (54), (56) and (58) yield

$$\int^y a_{3k+3}(u)H_{6m+s-3k-2}(u)H_1(u)^{-6m-s}\,\mathrm{d}u$$

$$= \begin{cases} \displaystyle\int^y \frac{(1-b_{01}u)^{2k+1}}{u^{4k+3}}\frac{P_{5m-\frac{5k}{2}+[\frac{s-3}{2}]}(u)}{u^{8m-4k+s-3}}\,\mathrm{d}u = \frac{P_{5m-\frac{k}{2}+[\frac{s-3}{2}]+1}(y)}{y^{8m+s-1}} \\ \text{when} \quad s=3,4,5 \\ \displaystyle\int^y \frac{(1-b_{01}u)^{2k+1}}{u^{4k+3}}\frac{P_{5m-\frac{5k}{2}+2}(u)}{u^{8m-4k+s-3}}\,\mathrm{d}u = \frac{P_{5m-\frac{k}{2}+3}(y)}{y^{8m+6}} \\ \text{when} \quad s=6. \end{cases}$$

In the case $s = 1, 2$ we obtain

$$-\int^y a_{3k+3}(u)H_{6(m-\frac{k}{2}-1)+s+4}(u)H_1(u)^{-6m-s}\,\mathrm{d}u = \int^y \frac{P_{5m-4+[\frac{s+4}{2}]}(u)}{u^{8m+s-1}} = \frac{P_{5m-3+[\frac{s+4}{2}]}(y)}{y^{8m+s-1}}.$$

All obtained formulae agree with (54).

Similarly, for k odd

$$\int^y a_{3k+3}(u)H_{6m+s-3k-2}(u)H_1(u)^{-6m-s}\,\mathrm{d}u = \int^y a_{3k+3}(u)H_{6(m-\frac{k+1}{2})+s+1}(u)H_1(u)^{-6m-s}\,\mathrm{d}u$$

$$= \begin{cases} \displaystyle\int^y \frac{(1-b_{01}u)^{2k+1}}{u^{4k+3}}\frac{P_{5m-\frac{5k}{2}-\frac{5}{2}+[\frac{s+2}{2}]}(u)}{u^{8m-4k+s-3}}\,\mathrm{d}u = \frac{P_{5m-\frac{k}{2}-\frac{3}{2}+[\frac{s+2}{2}]}(y)}{y^{8m+s-1}} \\ \text{when} \quad s=3,4,5 \\ \displaystyle\int^y \frac{(1-b_{01}u)^{2k+1}}{u^{4k+3}}\frac{P_{5m-\frac{5k}{2}+\frac{5}{2}}(u)}{u^{8m-4k+4}}\,\mathrm{d}u = \frac{P_{5m-\frac{k}{2}+\frac{7}{2}}(y)}{y^{8m+6}} \\ \text{when} \quad s=6. \end{cases}$$

And for $s = 1, 2$

$$-\int^y a_{3k+3}(u)H_{6(m-\frac{k+1}{2}-1)+s+1}(u)H_1(u)^{-6m-s}\,\mathrm{d}u = \int^y \frac{P_{5m-\frac{k}{2}-\frac{3}{2}}(u)}{u^{8m+s-1}} = \frac{P_{5m}(y)}{y^{8m+s-1}}.$$

Again, in agreement with (54). Analogously, one can consider the cases $l = 1$ and 2 and to check that (54) also holds in these cases. Therefore the system has a first integral of the form (50) with the coefficients H_k given by (54). According to proposition 2 from [11] it follows that there is also a Lyapunov integral of the form (12). Hence the system is linearizable by the substitution

$$z_1 = x l_1^{\alpha_1} l_2^{\alpha 2} \qquad z_2 = \frac{\Psi(x,y)}{x l_1^{\alpha_1} l_2^{\alpha 2}}$$

where l_1, l_2 are defined by (45), α_1, α_2 by (46) and Ψ is a Lyapunov integral (12).

(3) In this case the linearizing change is

$$z_1 = x l_1^{\alpha_3} l_2^{\alpha_2} l_3^{\alpha_1} \qquad z_2 = y l_1^{\alpha_3} l_2^{\alpha_2} l_3^{\alpha_1}$$

where

$$l_1 = 1 + \tfrac{1}{2}\left(-a_{10} - \sqrt{a_{10}^2 + 4\,a_{20}}\right)x + \tfrac{1}{2}\left(-b_{01} - \sqrt{b_{01}^2 + 4\,b_{02}}\right)y$$

$$l_2 = 1 + \tfrac{1}{2}\left(-a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}\right)x + \tfrac{1}{2}\left(-b_{01} - \sqrt{b_{01}^2 + 4\,b_{02}}\right)y$$

$$l_3 = 1 + \tfrac{1}{2}\left(-a_{10} - \sqrt{a_{10}^2 + 4\,a_{20}}\right)\,x + \tfrac{1}{2}\left(-b_{01} + \sqrt{b_{01}^2 + 4\,b_{02}}\right)y$$

and

$$\alpha_1 = -\frac{b_{01} + \sqrt{b_{01}^2 + 4\,b_{02}}}{2\sqrt{b_{01}^2 + 4\,b_{02}}} \qquad \alpha_2 = -\frac{-a_{10} + \sqrt{a_{10}^2 + 4\,a_{20}}}{2\sqrt{a_{10}^2 + 4\,a_{20}}}$$

$$\alpha_3 = \frac{b_{01}}{2\sqrt{b_{01}^2 + 4\,b_{02}}} - \frac{a_{10}}{2\sqrt{a_{10}^2 + 4\,a_{20}}}.$$

(X) Consider case (2) when the system has the form

$$\dot{x} = x(1 - a_{10}x) \qquad \dot{y} = -y(1 - b_{01}y - b_{20}x^2 - b_{3,-1}x^3y^{-1}) \tag{59}$$

and the equation of trajectories is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sum_{i=0}^{\infty} a_i x^i$$

where

$$a_n = \frac{P_{[(n-1)/3]}}{y^{[\frac{n-1}{3}+1]}\beta^{[\frac{n+1}{2}]}} \tag{60}$$

where P_m is a polynomial of degree m and $\beta = (1 - b_{01}y)$.

As in the case of system (47) we look for a first integral of the form (50). Then

$$H_1 = \frac{b_{01}y - 1}{y}$$

and one can prove by induction that

$$H_{2k}(y) = \frac{H_1(y)^{2k}P_{2k-2}(y)}{(y-1)^{3k}} \qquad H_{2k+1}(y) = \frac{H_1(y)^{2k}P_{2k-1}(y)}{(y-1)^{3k+1}}.$$

Hence, according to [11] the system has a centre in the origin, which is linearizable by the substitution

$$z_1 = \frac{x}{1 - a_{10}x} \qquad z_2 = \frac{\Psi(x, y)(1 - a_{10}x)}{x}.$$

□

Remark. In the case of the systems $a_{10} = a_{-12} = a_{20} = 0$ and $a_{01} = a_{-12} = a_{11} = 0$ using our computer facilities we were not able to find the primary decompositions of the corresponding ideals and, therefore, to find the necessary conditions of linearizability. So the problem for these systems is still open.

5. A linear operator dual to $D(\Psi)$

Let

$$W = \sum W_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a_{\bar{i}_1}^{\nu_1} a_{\bar{i}_2}^{\nu_2} \ldots a_{\bar{i}_l}^{\nu_l} b_{\bar{j}_l}^{\nu_{l+1}} b_{\bar{j}_{l-1}}^{\nu_{l+2}} \ldots b_{\bar{j}_1}^{\nu_{2l}} \tag{61}$$

be a formal series with $W(\bar{0}) = 1$. Let us denote

$$\mathcal{A}(W) = \sum_{(i,j)\in S} \frac{\partial W}{\partial a_{ij}} a_{ij}(i - j - i(1,a) + j(1,b)) + \sum_{(j,i)\in S} \frac{\partial W}{\partial b_{ij}} b_{ij}(i - j - i(1,a) + j(1,b)) \tag{62}$$

where $(1,a) = \sum_{(i,j)\in S} a_{ij}$, $(1,b) = \sum_{(j,i)\in S} b_{ij}$.

We call a function $f(a,b) \stackrel{\text{def}}{=} f(a_{\bar{\imath}_1}, a_{\bar{\imath}_2}, \ldots, a_{\bar{\imath}_l}, b_{\bar{\jmath}_l}, \ldots, b_{\bar{\jmath}_1})$ with the property

$$\mathcal{A}(f(a,b)) = k(a,b)f(a,b) \tag{63}$$

where $f(a,b)$ and $k(a,b)$ are polynomials, the eigenfunction of the operator $\mathcal{A}$, and the function $k(a,b)$ we call the cofunction.

Theorem 4. *(1) System (11) has a centre in the origin for all values of the parameters a_{kn}, b_{nk} if and only if there is a formal series (61) satisfying the equation*

$$\mathcal{A}(W) = W((1,a)-(1,b)). \tag{64}$$

(2) The origin is an isochronical centre of system (11) for all values of parameters a_{kn}, b_{nk} if and only if there are formal series $\tilde{W}, \hat{W}$ of the form (61) satisfying the equation

$$\mathcal{A}(\tilde{W}) = \tilde{W}(1,a) \qquad \mathcal{A}(\hat{W}) = -\hat{W}(1,b). \tag{65}$$

Proof. Let $F(u_1,\ldots,u_n) = \sum F_{(\nu_1,\ldots,\nu_n)} u_1^{\nu_1} \ldots u_n^{\nu_n}$ be a generating function. Then with the monomial $\nu_i F_{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_n)} u_1^{\nu_1} \ldots u_n^{\nu_n}$ one can associate the differential operator $u_i(u_i F)'_{u_i}$ when $i = j$ and the operator $u_i u_j F'_{u_i}$ if $i \neq j$ (see, e.g. [14]). Using these relations we get from (26) (taking into account theorem 2) the formulae (64) and (65), correspondingly. □

Corollary 3. *If there are formal series of the form (61), which are the solutions to equation (64) and to one of equations (65), then there is a formal series of the form (61), which is the solution to the other equation (65).*

The corollary provides a way of linearization analogous to the one given by the formulae (42), (43).

We have a hypothesis that there is a ring $\mathcal{P}$ of some functions of a, b such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}[[x,y]] & \xrightarrow{\pi} & \mathcal{P} \\ D\downarrow & & \downarrow\mathcal{A} \\ \mathcal{P}[[x,y]] & \xrightarrow{\pi} & \mathcal{P} \end{array} \tag{66}$$

where π is an isomorphism defined by

$$\pi : \sum c_{\alpha,\beta}(a,b)x^\alpha y^\beta \longrightarrow \sum c_{\alpha,\beta}(a,b) \tag{67}$$

and $D(f)$ is the operator (13).

We cannot give the complete description of the ring $\mathcal{P}$ but we consider some examples which show that the operator $\mathcal{A}(W)$ is a kind of dual operator to $D(f)$ in the sense that we can replace the search for a Lyapunov first integral by searching for a solution W of equation (64), the search for a linearizing transformation by the search for solutions of equations (65), and instead of looking for algebraic invariant curves we can seek solutions $f(a,b)$ of equation (63). This means, having solutions of (63)–(65) we obtain, using π^{-1}, the algebraic curve of system (11), the Lyapunov first integral or the linearizing transformation, respectively.

In the examples below $\mathcal{P}$ is the set of formal series constructed as follows. Consider an operator similar to (21), but more general. Namely, we allow the coordinates ν_k of the vector $\bar{\nu}$ to be some rational numbers. We denote such an operator by $\bar{L}(\nu)$ and by Ξ the monoid of all solutions of the equation

$$\bar{L}(\nu) = \binom{m}{n}$$

where m, n runs through the whole set of non-negative integers. Let $k[[\Xi]]$ be the monoid ring of the formal series of the monoid Ξ over the field k, which means the set of the formal series of the form

$$F = \sum \alpha_{(\nu)}[\nu]$$

where $\alpha_{(\nu)} \in k$, $\nu \in \Xi$. In the examples below we will see that if $f(a, b) \in \mathcal{P}$ is a solution of (63) then $\pi^{-1}(f)$ is an algebraic invariant curve, if $f = 1+\sum d_{\alpha,\beta}(a, b)$ is a solution of (64), then $xy\pi^{-1}(f)$ is a Lyapunov first integral and if f is a solution of the first (the second) of equations (65), then the first of the equations of system (11) is linearized by the substitution

$$z_1 = x\pi^{-1}(f)$$

(the second one by $z_2 = y\pi^{-1}(f)$) and the statements are reversible.

Consider now from this point of view system IV from table 1. In case (1) the operator (62) admits the eigenfunction

$$w_1 = 1 - a_{02}$$

with the cofunction

$$k_1 = 2b_{02}$$

and in this case $(1, b) = b_{02}$. Therefore

$$\hat{W} = (1 - a_{02})^{-1/2}$$

is a solution of the second of equations (65). In this case the symmetry conditions are fulfilled [18, 20, 31], hence there is a Lyapunov first integral $\Psi(x, y)$. This yields that the second equation of the system is linearizable by $z_2 = y\pi^{-1}(\hat{W})$ and the first one by $z_1 = \Psi(x, y)/(y\pi^{-1}(\hat{W}))$.

In case (2) the system has the form

$$\dot{x} = x(1 - a_{-12}x^{-1}y^2 - a_{20}x^2 + b_{02}y^2) \qquad \dot{y} = -y(1 + a_{20}x^2 - b_{02}y^2). \tag{68}$$

The operator (62) is

$$\begin{aligned}\mathcal{A}_{(IV)-(2)}(W) = {} & \frac{\partial W}{\partial a_{-12}} a_{-12}(-3 + a_{-12} + a_{20} - b_{02}) + \frac{\partial W}{\partial a_{20}} a_{20}(2 - 2a_{-12} - 2a_{20} + 2b_{02}) \\ & + \frac{\partial W}{\partial b_{02}} b_{02}(-2 - 2a_{20} + 2b_{02})\end{aligned} \tag{69}$$

and the corresponding operator $\bar{L}$ is

$$\bar{L}(\nu) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \nu_1 + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \nu_2 + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \nu_3$$

where $\nu_1 \in \mathbb{Z}$, $\nu_2, \nu_3 \in \mathbb{Q}$ and are of the form $|\nu_2| = \frac{\alpha}{2^r}$, $|\nu_3| = \frac{\beta}{2^r}$ with $\alpha, \beta, r \in \mathbb{N}$.

When $b_{02} \neq \pm a_{-12}\sqrt{a_{20}}$ we have four eigenfunctions $w_i = \pi(s_i)$ (where s_i are the invariant curves), namely,

$$\begin{aligned} & w_1 = 1 - \sqrt{a_{20}} - \sqrt{-a_{-12}\sqrt{a_{20}} + b_{02}} && w_2 = 1 - \sqrt{a_{20}} + \sqrt{-a_{-12}\sqrt{a_{20}} + b_{02}} \\ & w_3 = 1 + \sqrt{a_{20}} - \sqrt{a_{-12}\sqrt{a_{20}} + b_{02}} && w_4 = 1 + \sqrt{a_{20}} + \sqrt{a_{-12}\sqrt{a_{20}} + b_{02}}. \end{aligned} \tag{70}$$

The corresponding cofunctions are

$$\begin{aligned} k_1 &= -\sqrt{a_{20}} - a_{20} + b_{02} + \sqrt{-a_{-12}\sqrt{a_{20}} + b_{02}} \\ k_2 &= -\sqrt{a_{20}} - a_{20} + b_{02} - \sqrt{-a_{-12}\sqrt{a_{20}} + b_{02}} \\ k_3 &= \sqrt{a_{20}} - a_{20} + b_{02} + \sqrt{a_{-12}\sqrt{a_{20}} + b_{02}} \\ k_4 &= \sqrt{a_{20}} - a_{20} + b_{02} - \sqrt{a_{-12}\sqrt{a_{20}} + b_{02}}. \end{aligned} \tag{71}$$

Consider now the equation

$$\sum_{i=1}^{4} \alpha_i k_i = -(1, b). \tag{72}$$

The equation has the solution $\alpha_i = -1/4$ ($i = 1, 2, 3, 4$). Therefore the function

$$Z = -\tfrac{1}{4}\left(\sum_{i=1}^{4} \log w_i\right)$$

is the solution of the equation

$$\mathcal{A}_{(IV)-(2)}(Z) = -(1, b)$$

and, hence,

$$W_2 = \exp(Z) = (w_1 w_2 w_3 w_4)^{-1/4}$$

is the solution of the equation

$$\mathcal{A}_{(IV)-(2)}(W) = -(1, b)W.$$

This means that the second equation of (66) is linearizable by

$$z_2 = y\pi^{-1}(W_2).$$

So, in this case we were lucky to find the linearizing transformation for the second equation of system (68) using equation (72) just because there are some constants, α_i, satisfying the equation. However, the general situation is more complicated.

If we consider a system which has a Darboux integral or a Darboux linearization of the form

$$f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_s^{\alpha_s}$$

then the exponents α_i are, generally speaking, functions of the coefficients a_{ij}, b_{ji} of our system. Therefore, noting that for w_i of the form $w_i = 1 + h.o.t$ the property $\mathcal{A}(w_i) = k_i w_i$ yields

$$\mathcal{A}(\log w_i) = k_i$$

we see that an analogue of equation (36) is the equation

$$\sum_{i=1}^{s} \alpha_i k_i + \sum_{i=1}^{s} \mathcal{A}(\alpha_i) \log(w_i) = 0 \tag{73}$$

(if we look for a first integral of the form $1 + \sum_{i=1}^{\infty} h_i(x, y)$ with $h_i(x, y)$ being homogeneous polynomials of degree i) or

$$\sum_{i=1}^{s} \alpha_i k_i + \sum_{i=1}^{s} \mathcal{A}(\alpha_i) \log(w_i) = (1, a) - (1, b) \tag{74}$$

(if we look for a Lyapunov first integral) and an analogue of equation (38) is

$$\sum_{i=1}^{s} \alpha_i k_i + \sum_{i=1}^{s} \mathcal{A}(\alpha_i) \log(w_i) = (1, a). \tag{75}$$

One can check that in the case under consideration the equation

$$\sum_{i=1}^{4} \alpha_i k_i + \sum_{i=1}^{4} \mathcal{A}_{(IV)-(2)}(\alpha_i) \log(w_i) = 0 \tag{76}$$

(with w_i given by (70) and k_i by (71)) has the solution

$$\alpha_1 = \sqrt{\frac{a_{-12}\sqrt{a_{20}} + b_{02}}{-a_{-12}\sqrt{a_{20}} + b_{02}}} \qquad \alpha_2 = -\alpha_1 \qquad \alpha_3 = -1 \qquad \alpha_4 = 1.$$

Therefore the function

$$U_0 = w_1^{\alpha_1} w_2^{\alpha_2} w_3^{-1} w_4$$

is a solution to the equation $\mathcal{A}_{(IV)-2)}(W) = 0$,

$$W_0 = \frac{U_0 - 1}{-4\sqrt{a_{-12}a_{20}^{3/2} + b_{02}a_{20}}}$$

gives a solution to $\mathcal{A}_{(IV)-2)}(W) = ((1,a) - (1,b))W$ and the function

$$W_1 = W_0\,(w_1 w_2 w_3 w_4)^{1/4}$$

provides a solution to $\mathcal{A}_{(IV)-2)}(W) = (1,a)W$. Hence, the linearizing substitution is

$$z_1 = x\pi^{-1}(W_1).$$

In the case $b_{02} = a_{-12}\sqrt{a_{20}}$, the function

$$W_0 = \frac{-1}{2\,a_{20}^{1/2}(-1 + \sqrt{a_{20}}\,)} + \big(\sqrt{2}\,\big(\log\big(1 + \sqrt{a_{20}} - \sqrt{2}\,\sqrt{a_{-12}}\,a_{20}^{\frac{1}{4}}\,\big)$$
$$- \log\big(1 + \sqrt{a_{20}} + \sqrt{2}\,\sqrt{a_{-12}}\,a_{20}^{\frac{1}{4}}\,\big)\big)\big)/\big(8\sqrt{a_{-12}}a_{20}^{\frac{3}{4}}\big)$$

is a solution of the equation

$$\mathcal{A}_{(IV)-(\tilde{2})}(W) = ((1,a) - (1,b))W$$

where

$$\mathcal{A}_{(IV)-(\tilde{2})}(W) = \frac{\partial W}{\partial a_{-12}}a_{-12}\big(-3 + a_{-12} + a_{20} - a_{-12}\sqrt{a_{20}}\big)$$
$$+\frac{\partial W}{\partial a_{20}}a_{20}\big(2 - 2a_{-12} - 2a_{20} + 2a_{-12}\sqrt{a_{20}}\big). \qquad (77)$$

Also in this case

$$W_2 = w_1^{-\frac{1}{2}}(w_3 w_4)^{-1/4}$$

is a solution to

$$\mathcal{A}_{(IV)-(\tilde{2})}(W) = -(1,b)W.$$

Therefore the linearizing substitution is

$$z_1 = x\pi^{-1}(W_0)/\pi^{-1}(W_2) \qquad z_2 = y\pi^{-1}(W_2).$$

The case $b_{02} = -a_{-12}\sqrt{a_{20}}$ is similar.

Consider now system (IV)-(5). It is easily seen that

$$w = 1 + 2b_{20} - 4a_{-12}b_{20} + a_{-12}^2 b_{20}$$

is an eigenfunction of the operator $\mathcal{A}(W)$. One can also check that the function

$$W_0 = \big(1 - \tfrac{1}{3}a_{-12}\big)w^{-3/4}$$

is a solution to $\mathcal{A}(W) = (1,a) - (1,b)$ and

$$W_2 = w^{-1/4}$$

gives a solution of $\mathcal{A}(W) = -(1, b)$. Therefore the function $F = xy\pi^{-1}(W_0)$ is a first integral of system (IV)-(5) and the system is linearizable by

$$z_1 = x\pi^{-1}(W_0)/\pi^{-1}(W_2) \qquad z_2 = y\pi^{-1}(W_2).$$

Case (6) is a partial case of the system with homogeneous cubic nonlinearities [7]. Thus we have completed the consideration of case (IV) of theorem 3.

Here we would like to mention a problem we faced. The operator $\mathcal{A}(W)$ for the general cubic system (31) has the form

$$\sum_{i+j=2}^{3} \frac{\partial W}{\partial a_{ij}} a_{ij}(i - j - i(1, a) + j(1, b)) + \sum_{i+j=2}^{3} \frac{\partial W}{\partial b_{ij}} b_{ij}(i - j - i(1, a) + j(1, b)) \tag{78}$$

Going back to the case (IV)-(2) we see that after substitution into the operator (78) the defining equations of case (IV)-(2), $b_{2,-1} = a_{02} + b_{02} = a_{20} + b_{20} = b_{11} = a_{11} = 0$, we get the operator (69). However, frankly speaking, we cannot claim, based solely on theorem 4 that if there is a solution of the equation

$$\mathcal{A}_{(IV)-(2)}(W) = W((1, a) - (1, b))$$

then the system has a centre, because we do not know of any proof of the statement that, after substituting into operator $\mathcal{A}(W)$ the defining equations of a component of the centre variety and after getting an operator denoted by $\bar{\mathcal{A}}$, an analogue of the theorem 4 holds. Nevertheless in such a case one can consider from the beginning system (68) and then easily derive an analogue of theorem 2, and from them an analogue of theorem 4.

The examples above show that in the case when the eigenfunctions w_i are polynomials and α_i are constants in order to get the linearizing transformation one can—with the same success—look for invariant algebraic curves or eigenfunctions of the equation (61) and then construct a linearizing transformation or a solution of the equations (65) (a first integral or a solution of (64)), correspondingly. However, if w_i are not polynomials or α_i are functions, then the problem of constructing the solutions of equations (63)–(65) becomes very difficult and it is preferable to apply the usual Darboux integration (linearization) method.

Nevertheless, we found one case where our new approach based on making use of the dual operator $\mathcal{A}(W)$ turned out to be more efficient than the traditional one. Using the approach for system VII-(1), i.e.

$$\dot{x} = x(1 - a_{01}y + a_{01}b_{10}x^2) \qquad \dot{y} = -y(1 - b_{10}x + a_{01}b_{10}y^2)$$

we obtained in [27] a type of linearizing transformation which was unknown before.

Consider this case in detail. Here the first of equations (65) has the form

$$\mathcal{A}(W) = W(a_{01} + b_{10}^2 - a_{01}b_{10}) \tag{79}$$

where

$$\mathcal{A}(W) = \frac{\partial W}{\partial a_{01}} a_{01}(-1 + b_{10} + a_{01}^2 - a_{01}b_{10}) + \frac{\partial W}{\partial b_{10}} b_{10}(1 - a_{01} - b_{10}^2 + a_{01}b_{10}).$$

Then $w_1 = 1 - a_{01}$, $w_2 = 1 - b_{10}$ are the eigenfunctions of the operator $\mathcal{A}(W)$ with the cofunctions

$$k_1 = a_{01} + a_{01}^2 - a_{01}b_{10} \qquad k_2 = -b_{10} - b_{10}^2 + a_{01}b_{10}$$

correspondingly.

If we assume that the Darboux exponents α_i are constants and try to construct a Darboux linearizing substitution using only these eigenfunctions, we have

$$\alpha_1 k_1 + \alpha_2 k_2 = a_{01} + b_{10}^2 - a_{01}b_{10}.$$

Obviously there are no α_1, α_2 satisfying this equation. But as we have mentioned above the exponents α_i of the function (34) are not necessarily constants but can also be functions of a_{ij}, b_{ji} satisfying, in our case, the equation

$$\begin{aligned}\alpha_1\pi^{-1}(k_1) + \alpha_2\pi^{-1}(k_2) - \pi^{-1}((1,a)) &= \alpha_1(a_{01}y + a_{01}^2y^2 - a_{01}b_{10}xy)\\ &\quad+\alpha_2(-b_{10}x - b_{10}^2x^2 + a_{01}b_{10}xy) + P(x,y)/x \equiv 1\end{aligned} \tag{80}$$

or

$$\begin{aligned}\alpha_1\pi^{-1}(k_1) + \alpha_2\pi^{-1}(k_2) + \pi^{-1}((1,b)) &= \alpha_1(a_{01}y + a_{01}^2y^2 - a_{01}b_{10}xy)\\ &\quad+\alpha_2(-b_{10}x - b_{10}^2x^2 + a_{01}b_{10}xy) + Q(x,y)/y \equiv -1.\end{aligned} \tag{81}$$

Still, it is easily seen that that there are no functions $\alpha_1(a,b)$, $\alpha_2(a,b)$, satisfying equations (80) or (81).

This means that the system is not Darboux linearizable in the sense of definition 6. Nevertheless we will show that in this case there exists a linearizing transformation in a form a bit more general than (34).

Indeed, in this case the symmetry conditions hold [18, 20, 31] and, therefore, the system has a centre in the origin. Hence, due to theorem 4 there is a formal series W_0 of the form (61) such that

$$A(W_0) = (a_{01} + b_{10}^2 - a_{01}^2 - b_{10})W_0.$$

Therefore we can consider the equation

$$\alpha_1k_1 + \alpha_2k_2 + \gamma(a_{01} + b_{10}^2 - a_{01}^2 - b_{10}) = a_{01} + b_{10}^2 - a_{01}b_{10}$$

which has the solution

$$\alpha_1 = \gamma = \tfrac{1}{2} \qquad \alpha_2 = -\tfrac{1}{2}.$$

It yields that for the function

$$Z = \tfrac{1}{2}\ln W_0 + \tfrac{1}{2}\ln(1-a_{01}) - \tfrac{1}{2}\ln(1-b_{10})$$

we have

$$\mathcal{A}(Z) = a_{01} + b_{10}^2 - a_{01}b_{10}$$

and, hence,

$$W = \exp Z = W_0^{1/2}(1-a_{01})^{1/2}(1-b_{10})^{-1/2}$$

satisfies (79).

Thus,

$$z_1 = \sqrt{\Psi}\left(\frac{x}{y}\right)^{1/2}\left(\frac{1-a_{01}y}{1-b_{10}x}\right)^{1/2} \qquad z_2 = \sqrt{\Psi}\left(\frac{y}{x}\right)^{1/2}\left(\frac{1-b_{10}x}{1-a_{01}y}\right)^{1/2} \tag{82}$$

is the linearizing substitution. Note that according to theorem 1 the coefficients v_{kn} of the integral Ψ are (k,n)-polynomials and, hence, the substitution is analytical.

The type of linearization given by (82) is more general than the type described in definition 6 and, to our knowledge, was unknown before.

6. Conclusions

In this paper we presented efficient algorithms for computing the focus and linearizability quantities for polynomial vector fields. We applied them to the investigation of the linearizability of eight-parametric subfamilies of the cubic system (31) with one quadratic and three cubic terms (per equation). There are 12 such subfamilies. For all these systems we computed, using Mathematica and the algorithm of theorem 2 the first 14 linearizability quantities, $i_{11}, j_{11}, \ldots, i_{77}, j_{77}$. Then for ten of these systems using Singular we found the primary decompositions of the corresponding ideals and, thus, obtained the necessary conditions of linearizability presented in table 1.

Then we proved that these conditions are also the sufficient ones, for all cases, except V-(9) and V-(6). Thus to complete the investigation of the linearizability problem for these subfamilies of cubic systems there remains to prove the sufficiency of the conditions V-(6), (9) and to find the necessary and sufficient conditions of the linearizability for the systems

$$\dot{x} = x(1 - a_{01}y - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3)$$
$$\dot{y} = -y(1 - b_{10}x - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy)$$

and

$$\dot{x} = x(1 - a_{01}y - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3)$$
$$\dot{y} = -y(1 - b_{10}x - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy).$$

We also have shown that there is a linear partial differential operator, $\mathcal{A}(W)$, which is dual to the Lie derivative of our vector field, in the sense that the isomorphism π, defined by (67), maps invariant curves on the phase plane to eigenfunctions of the operator $\mathcal{A}(W)$, first integrals to the solutions of the equation (64) and linearizing transformations to solutions of equations (65). Because π is an isomorphism, the statement is reversible. Using this new approach we found a new type of Darboux linearization, presented by formula (82), where both equations of the linearizing transformation contain the Lyapunov first integral.

Note also, that in fact in section 5 we presented a new method of constructing a partial solution of some first-order linear partial differential equations with polynomial coefficients. Namely, for some partial differential equations it is possible to find a dual second-order system of ordinary differential equations, such that the isomorphism (66) maps invariant curves and integrals of the ODE system to the solutions of our original PDE. Because integration of first-order PDEs is very a difficult problem and very few methods for its investigation are known, we believe it would be very useful to know for which classes of PDE our method can be applied. However, this could the subject of a separate paper.

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Appendix

We present in figure A.1 the Mathematica code for computing the focus and linearizability quantities for system (32) based on the algorithm of theorem 2. With obvious changes one

```
        (*The operator (12) for the system (32)*)

In[1]:= l1[nu1_,nu2_,nu3_,nu4_,nu5_,nu6_,nu7_,nu8_]:= 0 nu1 + 2 nu2 + 1 nu3  + 0 nu4 +
        2 nu5 + 1 nu6 + 0 nu7 + 1 nu8;
        l2[nu1_,nu2_,nu3_,nu4_,nu5_,nu6_,nu7_,nu8_]:= 1 nu1 + 0 nu2 + 1 nu3 + 2 nu4 +
        0 nu5 + 1 nu6 + 2 nu7+ 0 nu8;

        (* Set  a=b=1  to compute the focus quantities, a=1,b=0 to compute the isochronicity
        quantities i {kk}, and  a=0, b=1 to compute -j {kk}*)

In[3]:= a=1;  b=0;

        (*Definition of  the function (15)*)

In[4]:= v[k1_,k2_,k3_,k4_,k5_,k6_,k7_,k8_]:=v[k1,k2,k3,k4,k5,k6,k7,k8]=
        Module[{us,coef},coef=l1[k1,k2,k3,k4,k5,k6,k7,k8]-l2[k1,k2,k3,k4,k5,k6,k7,k8]; us=0;
        v[0,0,0,0,0,0,0,0]=1;
         If[k1>0,us=us+(l1[k1-1,k2,k3,k4,k5,k6,k7,k8]+a)*v[k1-1,k2,k3,k4,k5,k6,k7,k8]];
        If[k2>0,us=us+(l1[k1,k2-1,k3,k4,k5,k6,k7,k8]+a)*v[k1,k2-1,k3,k4,k5,k6,k7,k8]];
        If[k3>0,us=us+(l1[k1,k2,k3-1,k4,k5,k6,k7,k8]+a)*v[k1,k2,k3-1,k4,k5,k6,k7,k8]];
        If[k4>0,us=us+(l1[k1,k2,k3,k4-1,k5,k6,k7,k8]+a)*v[k1,k2,k3,k4-1,k5,k6,k7,k8]];
        If[k5>0,us=us-(l2[k1,k2,k3,k4,k5-1,k6,k7,k8]+b)*v[k1,k2,k3,k4,k5-1,k6,k7,k8]];
        If[k6>0,us=us-(l2[k1,k2,k3,k4,k5,k6-1,k7,k8]+b)*v[k1,k2,k3,k4,k5,k6-1,k7,k8]];
        If[k7>0,us=us-(l2[k1,k2,k3,k4,k5,k6,k7-1,k8]+b)*v[k1,k2,k3,k4,k5,k6,k7-1,k8]];
        If[k8>0,us=us-(l2[k1,k2,k3,k4,k5,k6,k7,k8-1]+b)*v[k1,k2,k3,k4,k5,k6,k7,k8-1]];
        If [coef!=0,   us=us/coef]; If [coef==0, gg[k1,k2,k3,k4,k5,k6,k7,k8]=us;  us=0];  us]

        (* Maximal number of the focus quantity to be computed*)

In[6]:= gmax=7;

        (*Computing the quantities q[1], q[2], ... up to the order "gmax"*)

In[7]:= Do[k= sc; num=k;  q[num]=0;
        For [i1=0,i1<=2 k,i1++ ,
        For[i2=0,i2<=(2 k-i1),i2++ ,
        For[i3=0,i3<=(2 k-i1-i2),i3++ ,
        For[i4=0,i4<=(2 k-i1-i2-i3),i4++,
        For[i5=0,i5<=(2 k-i1-i2-i3-i4),i5++,
        For[i6=0,i6<=(2 k-i1-i2-i3-i4-i5),i6++,
        For[i7=0,i7<=(2 k-i1-i2-i3-i4-i5-i6),i7++,
        For[i8=0,i8<=(2 k-i1-i2-i3-i4-i5-i6-i7),i8++,
        If[(l1[i1,i2,i3,i4,i5,i6,i7,i8]==k) &&(l2[i1,i2,i3,i4,i5,i6,i7,i8]==k),
        v[i1,i2,i3,i4,i5,i6,i7,i8];
        q[num]=q[num]+gg[i1,i2,i3,i4,i5,i6,i7,i8]TT[i1,i2,i3,i4,i5,i6,i7,i8]]]]]]]]]],
        {sc,1,gmax}]

        (*Output of the computed quantities: the variables on the right-hand side should
        correspond to the coefficients of the system*)

In[8]:= TT[l1_,l2_,l3_,l4_,l5_,l6_,l7_,l8_]:=a01^l1 a20^l2 a11^l3  a02^l4   b20^l5  b11^l6
        b02^l7  b10^l8

In[9]:=
        Do[Print[" q[", i, i, "]=",q[i]], {i,1,gmax}]
```

Figure A.1. Mathematica code to compute the focus and linearizability quantities for system (32).

can apply the code to compute the quantities for the other system from table 1 as well as for computing the quantities for any polynomial system.

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